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A Mathematical Study on a Unified Variational Principle with Two Constrained Parameters

Jong Soon Im*, Yung Myun Yoo*, Byung Chai Lee and Yong Ju Lee****

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In this paper, a parameterized variational principle based on a mixed functional obtained by a linear combination of the total potential energy functional, the modified Hellinger-Reissner functional, and the Hu-Washizu functional with two constrained parameters is proposed, and the mathematical characteristics of the variational equation of the principle are investigated for the analysis of boundary value problems in linear elasticity. It is first proved that the Euler-Lagrange equations of the variational equation is identical to the governing equations for the given problem. Then existence of the unique solution of the variational equation is systematically proved by showing that the energy bilinear form is weakly-coercive. As an application, the stress/strain smoothing can be obtained as a form of mixed FEM based on the variational equation.

Key Words: Unified Variational Principle, Mixed Finite Element Method(FEM), Existence of the Unique Solution, Equivalence of Mixed and Displacement Based FEM, Weakly-coercive

1. Introduction

Several variational principles were developed as basis of the finite element method (FEM) in linear elasticity (e. g., Washizu, 1982). Among them, the principle of minimum potential energy leads to a variational equation involving positive definite bilinear forms that guarantees such desirable mathematical properties as boundedness, completeness and convergence (e. g., Oden and Reddy, 1976). Therefore, displacement based finite elemat mothod using this principle have been widely adopted in analyzing linear elastic problems.

However, there are some inevitable shortcomings in displacement based FEM by treating only the displacement field as an independent variable. Typical examples are loss of accuracy in stress/strain obtained from the derivatives of the displacement field on the boundary, and locking phenomena in "constrained problems," such as nearly incompressible elastic problems and bending of plates and shells (e. g., Zienkiewicz, 1977; Stolarski and Belytschko, 1983). Hence, in displacement based FEM, much research has been carried out to numerically overcome these shortcomings (e. g., Oden and Brauchli, 1971; Hinton and Campbell, 1974; Cantin et al., 1978; Zienkiewicz et al., 1971; Belytschko et al., 1981; Stolarski et al., 1983).

On the other hand, mixed finite element methods which treat stress/strain fields as additional independent variables provide more flexibility in the formulation of the finite element models. The flexibility has been used to improve the finite element formulations but it also creates a need for additional care. One of the most important applications of mixed FEM is to provide indirectly a theoretical basis for several numerical techniques an displacementbased FEM (e. g.,

* R & D Division Korea Automotive Technology Institute 74, Yongjung-Ri, Pungse-Myon Chonan, Chungnam 330-910, Korea

** Department of Mechanical Engineering Korea Advanced Institute of Science and Technology 373-1, Guseong-Dong, Yuseong-Gu Deajeon 305-338, Korea

Malkus and Hughes, 1978; Shimodaira, 1984; Stolarski and Belytschko, 1986; Simo and Hughes, 1986). For example, the reduced integration technique had not been regarded as a valid method until the equivalence of a class of mixed models with reduced integration single-field elements in linear elastic problems was proved by Malkus and Hughes (1978).

While there are advantages to mixed FEM theory, there are theoretical difficulties not yet solved. First, the existence of the unique solutions of all mixed variational equations has not been proven. Therefore, it can happen that in mixed FEM the stiffness matrices become singular when certain shape functions are adopted (Mirza and Olson, 1980), which prohibited wide use of the method. Recently, several approaches were developed to cope with this difficulty. A numerical method was developed to convert the indefinite system into a positive definite system by premultiplying the transpose of the stiffness matrix to the stiffness matrix (Mirza, 1984). However, this significantly increases computational efforts. To avoid this problem, a generalized mixed variational principle based on a combined functional of the potential energy functional and the Hellinger-Reissner functional was proposed and mathematically shown that the variational equation of the principle involves positive definite bilinear forms (Slivker, 1984; Lee and Lee, 1990). However, strict constraints were imposed in the construction of the mixed functional for the positive definiteness of the energy bilinear form. The constraints in turn seriously restrict the development of various mixed models with good numerical performance.

Second, mixed FEM based on existing variational principles does not provide a complete theoretical basis for numerical techniques of displacement based FEM. This explains why most equivalences of the mixed models with single field models are satisfied numerically only when special shape functions in mixed FEM and special numerical integration algorithms in displacement based FEM are used (e. g., Malkus and Hughes, 1978; Shimodaira, 1984; Stolarski and Belytschko, 1986). To provide a more generalized equivalence theory of mixed FEM with displacement based FEM, a new unified variational principle needs to be developed.

To guarantee existence and uniqueness of solutions and to offer a basis for the development of other various mixed models, it is required that the new unified variational principle should provide variational equations with weakly-coercive, not necessarily positive definite, energy bilinear forms (e. g., Oden and Reddy, 1976). Felippa (1994) suggested a construction method for various parameterized functionals and their associated variational principle but did not investigate the mathematical aspects of the functionals such as the existence and uniqueness of solutions. Furthermore, to establish a more concrete variational basis for numerical techniques of displacement based FEM, it is desirable that one derive the basic variational equation for the techniques from the parameterized variational principle.

In this paper, a unified variational principle with two constrained parameters is proposed, and mathematical characteristics of the variational equation of the principle are investigated for analysis of the boundary value problems in linear elasticity. The principle is based on a mixed functional obtained by a linear combination of the total potential energy functional, the modified Helliger-Reissner functional (Lee and Pian, 1978) and the Hu-Washizu functional with two constrained parameters. As a necessary condition of the principle, it is first proved that the Euler-Lagrange equations of the variationals equation are identical to the governing equations for the given problem. Then existence of the unique solution of the variational equation is systematically proved by showing that the energy bilinear form is weakly-coercive. Applicability of the new unified variational principle is demonstrated by showing that displacement based FEM with the widely used stress/strain smoothing method of Hinton and Campbell (1974) can be obtained as a form of mixed FEM based on the variational equation.

2. A Unified Variational Principle with Two Constrained Parameters in Linear Elasticity

2.1 Boundary value problem in linear elasticity

Let the domain of an elastic body Ω be bounded in $\mathbf{R}^3$ by a Lipschitz boundary Γ. Denote by $u=[u_1, u_2, u_3]^T$ the displacement vector. The linear strain tensor is defined as

$$\varepsilon_{ij}(u)=u_{(i,j)}=\frac{1}{2}(u_{i,j}+u_{j,i}) \quad i,j=1,2,3 \tag{1}$$

where u_{ij}, and $u_{(i,j)}$ denote the partial derivative of u_i with respect to x_j and the symmetric gradient tensor of u, respectively. The stress-strain relationships are given as

$$\sigma_{ij}(u)=\sum_{k,l=1}^{3} C_{ijkl}\varepsilon_{kl}(u), \quad i,j=1,2,3 \tag{2}$$

$$\varepsilon_{ij}(u)=\sum_{k,l=1}^{3} D_{ijkl}\sigma_{kl}(u), \quad i,j=1,2,3 \tag{3}$$

where C_{ijkl} and D_{ijkl} denote the elastic modulus tensor and its inverse, respectively. The elastic modulus and its inverse tensors are assumed to be symmetric and positive definite as

$$C_{ijkl}=C_{jikl}=C_{klij}=C_{ijlk}, \quad i,j,k,l=1,2,3 \tag{4}$$

$$\mu_m\sum_{i,j=1}^{3}\varepsilon_{ij}\varepsilon_{ij}\leq\sum_{i,j,k,l=1}^{3}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \leq\mu_M\sum_{i,j=1}^{3}\varepsilon_{ij}\varepsilon_{ij} \tag{5}$$

$$\frac{1}{\mu_M}\sum_{i,j=1}^{3}\sigma_{ij}\sigma_{ij}\leq\sum_{i,j,k,l=1}^{3}D_{ijkl}\sigma_{ij}\sigma_{kl} \leq\frac{1}{\mu_m}\sum_{i,j=1}^{3}\sigma_{ij}\sigma_{ij} \tag{6}$$

where the minimum elastic modules μ_m and the maximum elastic modules μ_M are positive constants.

Denote by $f=[f_1, f_2, f_3]^T$ the body force vector in Ω. When the displacement is suppressed on Γ_0 and the traction $T=[T_1, T_2, T_3]^T$ is given on Γ_1, the state equations are expressed as

$$\sum_{j=1}^{3}\sigma_{ij,j}+f_i=0, \quad i=1,2,3 \text{ in } \Omega \tag{7}$$

$$u_i=0, \quad i=1,2,3 \text{ in } \Gamma_0 \tag{8}$$

$$\sum_{j=1}^{3}\sigma_{ij}n_j=T_i, \quad i=1,2,3 \text{ in } \Gamma_1 \tag{9}$$

where $n=[n_1, n_2, n_3]^T$ denotes the unit outward normal vector, $\Gamma=\Gamma_0\cup\Gamma_1$ and $\Gamma_0\cap\Gamma_1\neq\phi$.

2.2 A functional for the unified variational principle

For the mixed formulation treating displacement u, strain ε and stress σ as independent variables, one can define admissible sets for the given problem as

$$U=\{u=\{u_i\},\ u_i\in H^1(\Omega): u_i=0,\ i=1,2,3,\ x\in\Gamma_0\} \tag{10}$$

$$E=\{\varepsilon=\{\varepsilon_{ij}\},\ \varepsilon_{ij}\in H^0(\Omega): \varepsilon_{ij}=\varepsilon_{ji},\ i,j=1,2,3\} \tag{11}$$

$$S=\{\sigma=\{\sigma_{ij}\},\ \sigma_{ij}\in H^0(\Omega): \sigma_{ij}=\sigma_{ij},\ i,j=1,2,3\} \tag{12}$$

where $H^m(\Omega)$ denotes a Sobolev space of order m. U, E and S are Hilbert spaces with norms

$$\|u\|_1^2=\int_\Omega\sum_{i=1}^{3}\left\{u_i^2+\sum_{j=1}^{3}\left(\frac{\partial u_i}{\partial x_j}\right)^2\right\}d\Omega \tag{13}$$

$$\|\varepsilon\|_0^2=\int_\Omega\sum_{i,j=1}^{3}\varepsilon_{ij}\varepsilon_{ij}d\Omega \tag{14}$$

$$\|\sigma\|_0^2=\int_\Omega\sum_{i,j=1}^{3}\sigma_{ij}\sigma_{ij}d\Omega \tag{15}$$

where $\|\cdot\|_m^2$ denotes a square of the m-th order Sobolev norm.

Let V be the natural product space of U, E and S. Then V is also a Hilbert space with the natural product norm given by

$$\|(u,\varepsilon,\sigma)\|_V^2=\|u\|_1^2+\|\varepsilon\|_0^2+\|\sigma\|_0^2 \tag{16}$$

or, as an equivalent norm in the mathematical sense,

$$\|(u,\varepsilon,\sigma)\|'_V=\|u\|_1+\|\varepsilon\|_0+\|\sigma\|_0 \tag{17}$$

For the unified variational principle of the linear elastic problem, one can construct a three-field mixed functional on the space V defined as

$$J\equiv(1+\alpha+\beta)\left[\int_\Omega\frac{1}{2}\sum_{i,j=1}^{3}\left(\sum_{k,l=1}^{3}C_{ijkl}u_{(k,l)}\right)u_{(i,j)}d\Omega -\int_{\Gamma_1}\sum_{i=1}^{3}T_iu_id\Gamma-\int_\Omega\sum_{i=1}^{3}f_iu_id\Omega\right]$$

$$-\alpha\left[\int_\Omega\sum_{i,j=1}^{3}\sigma_{ij}u_{(i,j)}d\Omega-\frac{1}{2}\int_\Omega\sum_{i,j=1}^{3}\left(\sum_{k,l=1}^{3}D_{ijkl}\sigma_{kl}\right)\sigma_{ij}d\Omega\right.$$

$$-\int_{\Gamma_1}\sum_{i=1}^{3}T_iu_id\Gamma-\int_{\Omega}\sum_{i=1}^{3}f_iu_id\Omega\Big]$$
$$-\beta\Big[\int_{\Omega}\sum_{i,j=1}^{3}(\sum_{k,l=1}^{3}C_{ijkl}\varepsilon_{kl})\,u_{(i,j)}d\Omega$$
$$-\frac{1}{2}\int_{\Omega}\sum_{i,j=1}^{3}(\sum_{k,l=1}^{3}C_{ijkl}\varepsilon_{kl})\,\varepsilon_{ij}d\Omega$$
$$-\int_{\Gamma_1}\sum_{i=1}^{3}T_iu_id\Gamma-\int_{\Omega}\sum_{i=1}^{3}f_iu_id\Omega\Big] \tag{18}$$

where α and β are constants subject to

$$\alpha\beta\neq 0 \tag{19}$$
$$1+\min\{0,\alpha\}+\min\{0,\beta\}>0 \tag{20}$$

This mixed functional is obtained by a linear combination of the total potential energy functional, the modified Hellinger-Reissner functional (Lee and Pian, 1978) and the Hu-Washizu functional via the constrained parameters α and β.

3. Mathematical Characteristics of the Unified Variational Equation

3.1 Euler-Lagrange equations

Let $(\delta\mathbf{u},\delta\boldsymbol{\varepsilon},\delta\boldsymbol{\sigma})\in V$ be the admissible variation of $(\mathbf{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma})\in V$. Then the first variational equation of J is given as

$$0\equiv\Big[(1+\alpha+\beta)\int_{\Omega}\sum_{i,j=1}^{3}(\sum_{k,l=1}^{3}C_{ijkl}u_{(k,l)})\,\delta u_{(i,j)}d\Omega$$
$$-\alpha\int_{\Omega}\sum_{i,j=1}^{3}\sigma_{ij}\delta u_{(i,j)}d\Omega$$
$$-\beta\int_{\Omega}\sum_{i,j=1}^{3}(\sum_{k,l=1}^{3}C_{ijkl}\varepsilon_{kl})\,\delta u_{(i,j)d}\Omega$$
$$-\int_{\Gamma_1}\sum_{i=1}^{3}T_i\delta u_id\Gamma-\int_{\Omega}\sum_{i=1}^{3}f_i\delta u_id\Omega\Big]$$
$$+\alpha\int_{\Omega}\sum_{i,j=1}^{3}(\sum_{k,l=1}^{3}D_{ijkl}\sigma_{kl}-u_{(i,j)})\,\delta\sigma_{ij}d\Omega$$
$$+\beta\int_{\Omega}\sum_{i,j=1}^{3}(\sum_{k,l=1}^{3}C_{ijkl}(\varepsilon_{kl}-u_{(k,l)}))\,\delta\varepsilon_{ij}d\Omega \tag{21}$$

Applying Green's theorem to Eq. (21), one gets

$$0=-\int_{\Omega}\sum_{i=1}^{3}\Big[\sum_{j=1}^{3}\Big((1+\alpha+\beta)\sum_{k,l=1}^{3}C_{ijkl}u_{(k,l)}$$
$$-\alpha\sigma_{ij}-\beta\sum_{k,l=1}^{3}C_{ijkl}\varepsilon_{kl}\Big)_{,j}+f_i\Big]\delta u_id\Omega$$
$$+\int_{\Gamma_1}\sum_{i=1}^{3}\Big[\sum_{j=1}^{3}\Big((1+\alpha+\beta)\sum_{k,l=1}^{3}C_{ijkl}u_{(k,l)}$$
$$-\alpha\sigma_{ij}-\beta\sum_{k,l=1}^{3}C_{ijkl}\varepsilon_{kl}\Big)n_j-T_i\Big]\delta u_id\Gamma$$
$$+\int_{\Omega}\alpha\sum_{i,j=1}^{3}\Big(\sum_{k,l=1}^{3}D_{ijkl}\sigma_{kl}-u_{(i,j)}\Big)\delta\sigma_{ij}d\Omega$$
$$+\int_{\Omega}\beta\sum_{i,j=1}^{3}\Big(\sum_{k,l=1}^{3}C_{ijkl}(\varepsilon_{kl}-u_{(k,l)})\Big)\delta\varepsilon_{ij}d\Omega \tag{22}$$

Thus, the Euler-Lagrange equations of the functional are obtained as

$$\sum_{j=1}^{3}\Big((1+\alpha+\beta)\sum_{k,l=1}^{3}C_{ijkl}u_{(k,l)}-\alpha\sigma_{ij}-\beta\sum_{k,l=1}^{3}C_{ijkl}\varepsilon_{kl}\Big)_{,j}+f_i=0,\quad i=1,2,3\ \text{in}\ \Omega \tag{23}$$

$$\sum_{j=1}^{3}\Big((1+\alpha+\beta)\sum_{k,l=1}^{3}C_{ijkl}u_{(k,l)}-\alpha\sigma_{ij}-\beta\sum_{k,l=1}^{3}C_{ijkl}\varepsilon_{kl}\Big)n_j-T_i=0,\quad i=1,2,3\ \text{on}\ \Gamma \tag{24}$$

$$\alpha\Big(\sum_{k,l=1}^{3}D_{ijkl}\sigma_{kl}-u_{(i,j)}\Big)=0,\quad i,j=1,2,3\ \text{in}\ \Omega \tag{25}$$

$$\beta\sum_{k,l=1}^{3}C_{ijkl}(\varepsilon_{kl}-u_{(k,l)})=0,\quad i,j=1,2,3\ \text{in}\ \Omega \tag{26}$$

From Eqs. (23) through (26) and the boundary condition of U, one can get the same governing equations as Eqs. (1) through (3) and Eqs. (7) through (9).

3.2 Existence and uniqueness of solutions

Taking $(\boldsymbol{\nu},\mathbf{e},\boldsymbol{\tau})\in V$ instead of $(\delta\mathbf{u},\delta\boldsymbol{\varepsilon},\delta\boldsymbol{\sigma})\in V$, Eq. (21) can be expressed as

$$B(\mathbf{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma},\boldsymbol{\nu},\mathbf{e},\boldsymbol{\tau})=l(\boldsymbol{\nu}),\ \forall(\boldsymbol{\nu},\mathbf{e},\boldsymbol{\tau})\in V \tag{27}$$

where an energy bilinear form $B(\mathbf{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma},\mathbf{e},\boldsymbol{\tau})$ and a load functional $l(\boldsymbol{\nu})$ are defined as

$$B(\mathbf{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma},\boldsymbol{\nu},\mathbf{e},\boldsymbol{\tau})$$
$$\equiv(1+\alpha+\beta)\int_{\Omega}\sum_{i,j=1}^{3}\Big(\sum_{k,l=1}^{3}C_{ijkl}u_{(k,l)}\Big)\nu_{(i,j)}d\Omega$$
$$+\alpha\int_{\Omega}\sum_{i,j=1}^{3}\Big(\sum_{k,l=1}^{3}D_{ijkl}\sigma_{(k,l)}\Big)\tau_{ij}d\Omega$$
$$-\alpha\int_{\Omega}\sum_{i,j=1}^{3}\Big(\sigma_{ij}\nu_{(i,j)}+\tau_{ij}u_{(i,j)}\Big)d\Omega$$
$$+\beta\int_{\Omega}\sum_{i,j=1}^{3}\Big(\sum_{k,l=1}^{3}C_{ijkl}\varepsilon_{kl}\Big)e_{ij}d\Omega$$
$$-\beta\int_{\Omega}\sum_{i,j=1}^{3}\Big(\sum_{k,l=1}^{3}C_{ijkl}(\varepsilon_{kl}\nu_{(i,j)}+e_{kl}u_{(i,j)})\Big)d\Omega \tag{28}$$

$$l(\boldsymbol{\nu})\equiv\int_{\Gamma_1}\sum_{i=1}^{3}T_i\nu_id\Gamma+\int_{\Omega}\sum_{i=1}^{3}f_i\nu_id\Omega \tag{29}$$

Since C_{ijkl} and D_{ijkl} are symmetric, the energy bilinear form is also symmetric about its arguments.

For convenience define artificial tensors associated with the independent variables as

$$\varepsilon_{ij}^{*}=\sum_{k,l=1}^{3} D_{ijkl}\sigma_{kl}, \quad i,j=1,2,3 \tag{30}$$

$$\sigma_{ij}^{*}=\sum_{k,l=1}^{3} C_{ijkl}\varepsilon_{kl}, \quad i,j=1,2,3 \tag{31}$$

$$\sigma_{ij}^{**}=\sum_{k,l=1}^{3} C_{ijkl}u_{(k,l)}, \quad i,j=1,2,3 \tag{32}$$

Since the elastic modulus tensor and its inverse are symmetric and positive definite, one can get the following inequalities:

$$\int_{\Omega}\sum_{i,j=1}^{3}\sigma_{ij}\varepsilon_{ij}^{*}d\Omega=\int_{\Omega}\sum_{i,j,k,l=1}^{3} D_{ijkl}\sigma_{ij}\sigma_{kl}d\Omega \geq \frac{1}{\mu_M}\int_{\Omega}\sum_{i,j=1}^{3}\sigma_{ij}\sigma_{ij}d\Omega=\frac{1}{\mu_M}\|\sigma\|_0^2 \tag{33}$$

$$\int_{\Omega}\sum_{i,j=1}^{3}\sigma_{ij}\varepsilon_{ij}^{*}d\Omega=\int_{\Omega}\sum_{i,j,k,l=1}^{3} C_{ijkl}\varepsilon_{ij}^{*}\varepsilon_{kl}^{*}d\Omega \geq \mu_m\int_{\Omega}\sum_{i,j=1}^{3}\varepsilon_{ij}^{*}\varepsilon_{ij}^{*}d\Omega=\mu_m\|\varepsilon^{*}\|_0^2 \tag{34}$$

By Hölder's inequality (see, e. g., Oden and Reddy, 1976), one has

$$\int_{\Omega}\sum_{i,j=1}^{3}\sigma_{ij}\varepsilon_{ij}^{*}d\Omega\leq\|\sigma\|_0\|\varepsilon^{*}\|_0 \tag{35}$$

From Eqs. (33) through (35), one can get the following inequalities

$$\frac{1}{\mu_M}\|\sigma\|_0\leq\|\varepsilon^{*}\|_0\leq\frac{1}{\mu_m}\|\sigma\|_0 \tag{36}$$

In a similar way, one also can obtain the following inequalities

$$\mu_m\|\varepsilon\|_0\leq\|\sigma^{*}\|_0\leq\mu_M\|\varepsilon\|_0 \tag{37}$$

$$\mu_m\|\nabla_s u\|_0\leq\|\sigma^{**}\|_0\leq\mu_M\|\nabla_s u\|_0 \tag{38}$$

where $\nabla_n u=\{u_{(i,j)}\}$

[**Theorem 1**] The energy bilinear form $B(u,\varepsilon,\sigma,\nu,e,\tau)$ is continuous on V.

(Proof) For all $(u,\varepsilon,\sigma)\in V$ and $(\nu,e,\tau)\in V$, the following inequality holds by definition of $B(u,\varepsilon,\sigma,\nu,e,\tau)$ represented by Eq. (28):

$$\begin{aligned}|B(u,\varepsilon,\sigma,\nu,e,\tau)|\leq|1+\alpha+\beta| &\left|\int_{\Omega}\sum_{i,j=1}^{3}\left(\sum_{k,l=1}^{3}C_{ijkl}u_{(k,l)}\right)\nu_{(i,j)}d\Omega\right| \\ &+|\alpha|\left|\int_{\Omega}\sum_{i,j=1}^{3}\left(\sum_{k,l=1}^{3}D_{ijkl}\sigma_{kl}\right)\tau_{ij}d\Omega\right| \\ &+|\alpha|\left|\int_{\Omega}\sum_{i,j=1}^{3}(\sigma_{ij}\nu_{(i,j)}+\tau_{ij}u_{(i,j)})\,d\Omega\right| \\ &+|\beta|\left|\int_{\Omega}\sum_{i,j,k,l=1}^{3}C_{ijkl}\varepsilon_{kl}e_{ij}d\Omega\right| \\ &+|\beta|\left|\int_{\Omega}\sum_{i,j=1}^{3}\left(\sum_{k,l=1}^{3}C_{ijkl}(\varepsilon_{kl}\nu_{(i,j)}+e_{kl}u_{(i,j)})\right)d\Omega\right|\end{aligned} \tag{39}$$

Using Eqs. (30) through (32), one can reduce Eq. (39) to

$$\begin{aligned}|B(u,\varepsilon,\sigma,\nu,e,\tau)|\leq|1+\alpha+\beta| &\left|\int_{\Omega}\sum_{i,j=1}^{3}\sigma_{ij}^{**}\nu_{(i,j)}d\Omega\right| \\ &+|\alpha|\left|\int_{\Omega}\sum_{i,j=1}^{3}\varepsilon_{ij}^{*}\tau_{ij}d\Omega\right| \\ &+|\alpha|\left|\int_{\Omega}\sum_{i,j=1}^{3}(\sigma_{ij}\nu_{(i,j)}+\tau_{ij}u_{(i,j)})\,d\Omega\right| \\ &+|\beta|\left|\int_{\Omega}\sum_{i,j=1}^{3}\sigma_{ij}^{*}e_{ij}d\Omega\right| \\ &+|\beta|\left|\int_{\Omega}\sum_{i,j=1}^{3}(\sigma_{ij}^{**}e_{ij}+\tau_{ij}^{**}\varepsilon_{ij})\,d\Omega\right|\end{aligned} \tag{40}$$

Applying Hölder's inequality again to Eq. (40), one can obtain the following inequality

$$\begin{aligned}|B(u,\varepsilon,\sigma,\nu,e,\tau)|\leq&|1+\alpha+\beta|\,\|\sigma^{**}\|_0\|\nabla_s\nu\|_0 \\ &+|\alpha|\,\|\varepsilon^{*}\|_0\|\tau\|_0+|\alpha|(\|\sigma\|_0\|\nabla_s\nu\|_0 \\ &+\|\tau\|_0\|\nabla_s u\|_0) \\ &+|\beta|\,\|\sigma^{*}\|_0\|e\|_0+|\beta|(\|\sigma^{**}\|_0\|e\|_0+\|\tau^{**}\|_0\|\varepsilon\|_0)\end{aligned} \tag{41}$$

Using Eqs. (36) through (38), one can get the following inequality from Eq. (41):

$$\begin{aligned}|B(u,\varepsilon,\sigma,\nu,e,\tau)|\leq&|1+\alpha+\beta|u_M\|\nabla_s u\|_0\|\nabla_s\nu\|_0 \\ &+\frac{|\alpha|}{\mu_m}\|\sigma\|_0\|\sigma\|_0+|\alpha|(\|\sigma\|_0\|\nabla_s\nu\|_0+\|\tau\|_0\|\nabla_s u\|_0) \\ &+|\beta|\mu_M\|\varepsilon\|_0\|e\|_0+|\beta|\mu_M(\|\nabla_s u\|_0\|e\|_0+\|\nabla_s\nu\|_0\|\varepsilon\|_0) \\ \leq&\gamma(\|\nabla_s u\|_0+\|\varepsilon\|_0+\|\sigma\|_0)(\|\nabla_s\nu\|_0+\|e\|_0+\|\tau\|_0)\end{aligned} \tag{42}$$

where γ is a positive constant given as

$$\gamma=\max\left\{|1+\alpha+\beta|\mu_M,\frac{|\alpha|}{\mu_m},|\alpha|\mu_M,|\alpha|,|\beta|\mu_M\right\}>0 \tag{43}$$

Since

$$\|\nabla_s u\|_0\leq\|u\|_1, \tag{44}$$

one can obtain the following inequality from Eq. (42)

$$|B(u,\varepsilon,\sigma,\nu,e,\tau)|\leq\gamma(\|u\|_1+\|\varepsilon\|_0$$

$$+\|\sigma\|_0)(\|\nu\|_1+\|e\|_0+\|\tau\|_0) \quad (45)$$

Inequality (45) now gives the desired conclusion.

[**Theorem 2**] If the load functional $l(\nu)$ is continuous, there exists a unique solution of the variational equation given by Eq. (27).

(Proof) If it is shown that the energy bilinear form is at least V-Weakly coercive, the proof is complete by [Theorem 1] and the Generalized Lax-Milgram Theorem (see, e. g., Oden and Reddy, 1976). For this, one may first obtain a set of inequalities on the energy bilinear form for four cases according to the values of α and β as follows:

(i) $\alpha<0$, $\beta<0$, and $1+\alpha+\beta>0$

Replacing (ν, e, τ) in Eq. (28) by $(u, -\varepsilon, -\sigma)$, one has

$$|B(u, \varepsilon, \sigma, u, -\varepsilon, -\sigma)=(1+\alpha+\beta)\int_\Omega\sum_{i,j=1}^{3}\Big(\sum_{k,l=1}^{3}C_{ijkl}u_{(k,l)}\Big)u_{(i,j)}d\Omega -\alpha\int_\Omega\sum_{i,j=1}^{3}\Big(\sum_{k,l=1}^{3}D_{ijkl}\sigma_{kl}\Big)\sigma_{ij}d\Omega -\beta\int_\Omega\sum_{i,j=1}^{3}\Big(\sum_{k,l=1}^{3}C_{ijkl}\varepsilon_{kl}\Big)\varepsilon_{ij}d\Omega \quad (46)$$

Applying Eqs. (5) and (6) to Eq. (46), one gets

$$B(u, \varepsilon, \sigma, u, -\varepsilon, -\sigma)\geq(1+\alpha+\beta)\mu_m\|\nabla_s u\|_0^2-\frac{\alpha}{\mu_M}\|\sigma\|_0^2-\beta\mu_m\|\varepsilon\|_0^2 \quad (47)$$

Korn's inequality (see, e. g., Oden and Reddy, 1983) for this boundary value problem is given as

$$\|\nabla_s u\|_0>C_K\|u\|_1 \quad (48)$$

where C_K is a positive constant. Substituting Eq. (48) into Eq. (47), one can obtain the following desired inequality

$$B(u, \varepsilon, \sigma, u, -\varepsilon, -\sigma)\geq C_1(\|u\|_1^2+\|\varepsilon\|_0^2+\|\sigma\|_0^2)=C_1\|(u, \varepsilon, \sigma)\|_V^2 \quad (49)$$

where C_1 is a positive constant given as

$$C_1=\min\Big\{(1+\alpha+\beta)\mu_m C_K^2,\ -\frac{\alpha}{\mu_M},\ -\beta\mu_m\Big\}>0 \quad (50)$$

(ii) $\alpha<0$, $\beta>0$, and $1+\alpha>0$

Replacing (ν, e, τ) in Eq. (28) by $(u, \varepsilon, -\sigma)$ and using Eq. (30), one obtains

$$B(u, \varepsilon, \sigma, u, \varepsilon, -\sigma)=(1+\alpha)\int_\Omega\sum_{i,j=1}^{3}\Big(\sum_{k,l=1}^{3}C_{ijkl}u_{(k,l)}\Big)u_{(i,j)}d\Omega -\alpha\int_\Omega\sum_{i,j=1}^{3}\Big(\sum_{k,l=1}^{3}C_{ijkl}\varepsilon^*_{kl}\Big)\varepsilon^*_{ij}d\Omega +\beta\int_\Omega\sum_{i,j=1}^{3}\Big(\sum_{k,l=1}^{3}C_{ijkl}(\varepsilon_{kl}-u_{(k,l)})\Big)(\varepsilon_{ij}-u_{(i,j)})d\Omega \quad (51)$$

Applying Eq. (5) to Eq. (51), one gets

$$B(u, \varepsilon, \sigma, u, \varepsilon, -\sigma)\geq(1+\alpha)\mu_m\int_\Omega\sum_{i,j=1}^{3}u_{(i,j)}u_{(i,j)}d\Omega -\alpha\mu_m\int_\Omega\sum_{i,j=1}^{3}\varepsilon^*_{ij}\varepsilon^*_{ij}d\Omega +\beta\mu_m\int_\Omega\sum_{i,j=1}^{3}(\varepsilon_{ij}-u_{(i,j)})(\varepsilon_{ij}-u_{(i,j)})d\Omega \quad (52)$$

By Hölder's inequality, one has

$$\Big|\int_\Omega\sum_{i,j=1}^{3}\varepsilon_{ij}u_{(i,j)}d\Omega\Big|\leq\|\varepsilon\|_0\|\nabla_s u\|_0 \quad (53)$$

For any positive constant χ, one always has

$$\chi\Big(\|\varepsilon\|_0-\frac{1}{\chi}\|\nabla_s u\|_0\Big)^2\geq 0 \quad (54)$$

From Eqs. (53) and (54), one can obtain

$$2\Big|\int_\Omega\sum_{i,j=1}^{3}\varepsilon_{ij}u_{(i,j)}d\Omega\Big|\leq 2\|\varepsilon\|_0\|\nabla_s u\|_0 \leq\chi\|\varepsilon\|_0^2+\frac{1}{\chi}\|\nabla_s u\|_0^2 \quad (55)$$

Substituting Eq. (55) into Eq. (52), one gets

$$B(u, \varepsilon, \sigma, u, \varepsilon, -\sigma)\geq(1+\alpha)\mu_m\|\nabla_s u\|_0^2-\alpha\mu_m\|\varepsilon^*\|_0^2 +B\mu_m\Big[(1-\chi)\|\varepsilon\|_0^2+\Big(1-\frac{1}{\chi}\Big)\|\nabla_s u\|_0^2\Big] \geq\Big[1+\alpha+\beta\Big(1-\frac{1}{\chi}\Big)\Big]\mu_m\|\nabla_s u\|_0^2 -\alpha\mu_m\|\varepsilon^*\|_0^2+\beta\mu_m(1-\chi)\|\varepsilon\|_0^2 \quad (56)$$

Letting $1>\chi>\dfrac{\beta}{1+\alpha+\beta}$ and applying Eqs. (36) and (48) to Eq. (56), one can obtain

$$B(u, \varepsilon, \sigma, u, \varepsilon, -\sigma)\geq C_2(\|u\|_1^2+\|\varepsilon\|_0^2+\|\sigma\|_0^2)=C_2\|(u, \varepsilon, \sigma)\|_V^2 \quad (57)$$

where C_2 is a positive constant given as

$$C_2=\min\left\{\left[1+\alpha+\beta\left(1-\frac{1}{\chi}\right)\right]\mu_m C_K{}^2, -\alpha\frac{\mu_m}{\mu_M{}^2}, \beta(1-\chi)\mu_m\right\}>0 \qquad (58)$$

(iii) $\alpha>0$, $\beta<0$, and $1+\beta>0$

Replacing $(\boldsymbol{\nu}, \boldsymbol{e}, \boldsymbol{\tau})$ in Eq. (28) by $(\boldsymbol{u}, -\boldsymbol{\varepsilon}, \boldsymbol{\sigma})$ and using Eq. (30), one obtains

$$\begin{aligned}B(\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{u}, -\boldsymbol{\varepsilon}, \boldsymbol{\sigma})&=(1+\alpha)\int_\Omega\sum_{i,j=1}^{3}\left(\sum_{k,l=1}^{3}C_{ijkl}u_{(k,l)}\right)u_{(i,j)}d\Omega\\&+\alpha\int_\Omega\sum_{i,j=1}^{3}\left(\sum_{k,l=1}^{3}C_{ijkl}(\varepsilon^*_{kl}-u_{(k,l)})\right)(\varepsilon^*_{ij}-u_{(i,j)})\,d\Omega\\&-\beta\int_\Omega\sum_{i,j=1}^{3}\left(\sum_{k,l=1}^{3}C_{ijkl}\varepsilon_{kl}\right)\varepsilon_{ij}d\Omega\end{aligned} \qquad (59)$$

Substituting Eqs. (5) and (55) into Eq. (59), one gets

$$\begin{aligned}B(\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{u}, -\boldsymbol{\varepsilon}, \boldsymbol{\sigma})&\geq(1+\alpha)\mu_m\|\nabla_s\boldsymbol{u}\|_0{}^2\\&+\alpha\mu_m\left[(1-\chi)\|\boldsymbol{\varepsilon}^*\|_0{}^2+\left(1-\frac{1}{\chi}\right)\|\nabla_s\boldsymbol{u}\|_0{}^2\right]\\&-\beta\mu_m\|\boldsymbol{\varepsilon}\|_0{}^2\\&\geq\left[1+\alpha\left(1-\frac{1}{\chi}\right)+\beta\right]\mu_m\|\nabla_s\boldsymbol{u}\|_0{}^2\\&+\alpha\mu_m(1-\chi)\|\boldsymbol{\varepsilon}^*\|_0{}^2-\beta\mu_m\|\boldsymbol{\varepsilon}\|_0{}^2\end{aligned} \qquad (60)$$

Letting $1>\chi>\frac{\alpha}{1+\alpha+\beta}$ and applying Eqs. (36) and (48) to Eq. (60), one can obtain

$$\begin{aligned}B(\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{u}, -\boldsymbol{\varepsilon}, \boldsymbol{\sigma})&\geq C_3(\|\boldsymbol{u}\|_1{}^2+\|\boldsymbol{\varepsilon}\|_0{}^2+\|\boldsymbol{\sigma}\|_0{}^2)\\&=C_3\|(\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})\|_V^2\end{aligned} \qquad (61)$$

where C_3 is a positive constant given as

$$C_3=\min\left\{\left[1+\alpha\left(1-\frac{1}{\chi}\right)+\beta\right]\mu_m C_K{}^2, \alpha(1-\chi)\frac{\mu_m}{\mu_M{}^2}, -\beta\mu_m\right\}>0 \qquad (62)$$

(iv) $\alpha>0$ and $\beta>0$

Replacing $(\boldsymbol{\nu}, \boldsymbol{e}, \boldsymbol{\tau})$ in Eq. (28) by $(\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})$ and using Eq. (30), one obtains

$$\begin{aligned}B(\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})&=\int_\Omega\sum_{i,j=1}^{3}\left(\sum_{k,l=1}^{3}C_{ijkl}u_{(k,l)}\right)u_{(i,j)}d\Omega\\&+\alpha\int_\Omega\sum_{i,j=1}^{3}\left(\sum_{k,l=1}^{3}C_{ijkl}(\varepsilon^*_{kl}-u_{(k,l)})\right)(\varepsilon^*_{ij}-u_{(i,j)})\,d\Omega\\&\times\beta\int_\Omega\sum_{i,j=1}^{3}\left(\sum_{k,l=1}^{3}C_{ijkl}(\varepsilon_{kl}-u_{(k,l)})\right)(\varepsilon_{ij}-u_{(i,j)})\,d\Omega\end{aligned} \qquad (63)$$

Applying Eqs. (5) and (55) to Eq. (63), one gets

$$\begin{aligned}B(\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})&\geq\mu_m\|\nabla_s\boldsymbol{u}\|_0{}^2+\alpha\mu_m\left[(1-\chi)\|\boldsymbol{\varepsilon}^*\|_0{}^2+\left(1-\frac{1}{\chi}\right)\|\nabla_s\boldsymbol{u}\|_0{}^2\right]\\&+\beta\mu_m\left[(1-\chi)\|\boldsymbol{\varepsilon}\|_0{}^2+\left(1-\frac{1}{\chi}\right)\|\nabla_s\boldsymbol{u}\|_0{}^2\right]\\&\geq\left[1+\alpha\left(1-\frac{1}{\chi}\right)+\beta\left(1-\frac{1}{\chi}\right)\right]\mu_m\|\nabla_s\boldsymbol{u}\|_0{}^2\\&+\alpha\mu_m(1-\chi)\|\boldsymbol{\varepsilon}^*\|_0{}^2+\beta\mu_m(1-\chi)\|\boldsymbol{\varepsilon}\|_0{}^2\end{aligned} \qquad (64)$$

Letting $1>\chi>\frac{\alpha+\beta}{1+\alpha+\beta}$ and applying Eqs. (36) and (48) to Eq. (64), one can obtain

$$\begin{aligned}B(\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})&\geq C_4(\|\boldsymbol{u}\|_1{}^2+\|\boldsymbol{\varepsilon}\|_0{}^2+\|\boldsymbol{\sigma}\|_0{}^2)\\&=C_4\|(\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})\|_V\end{aligned} \qquad (65)$$

where C_4 is a positive constant given as

$$C_4=\min\left\{\left[1+\alpha\left(1-\frac{1}{\chi}\right)+\beta\left(1-\frac{1}{\chi}\right)\right]\mu_m C_K^2, -\alpha(1-\chi)\frac{\mu_m}{\mu_M{}^2}, \beta(1-\chi)\mu_m\right\}>0 \qquad (66)$$

From the results of (i) ~ (iv) and definitions of supremum and infimum (see, e. g., Johnsonbaugh and Pfaffenberger, 1981), one can see that there always exists a positive constant C_0 such that

$$\sup_{(\boldsymbol{\nu},\boldsymbol{e},\boldsymbol{\tau})\in V}|B(\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{\nu}, \boldsymbol{e}, \boldsymbol{\tau})|\geq C_0\|(\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})\|_V^2>0,$$
$$(\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})\in V,\ (\boldsymbol{\nu}, \boldsymbol{e}, \boldsymbol{\tau})\in V,\ \|(\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})\|_V\neq0 \qquad (67)$$

$$\inf_{\|(\boldsymbol{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma})\|_V=1}\ \sup_{\|(\boldsymbol{\nu},\boldsymbol{e},\boldsymbol{\tau})\|_V=1}|B(\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{\nu}, \boldsymbol{e}, \boldsymbol{\tau})|\geq\inf_{\|(\boldsymbol{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma})\|_V=1}C_0\|(\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})\|_V^2=C_0>0 \qquad (68)$$

Inequalities (67) and (68) demonstrate that the energy bilinear form $B(\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{\nu}, \boldsymbol{e}, \boldsymbol{\tau})$ is weakly-coercive (see, e. g., Oden and Reddy, 1976). Hence, the proof is complete. In case of (iv), the energy bilinear form $B(\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{\nu}, \boldsymbol{\varepsilon}, \boldsymbol{\tau})$ is indeed coercive, i. e., V-Elliptic as shown in Eqs. (65) and (66).

4. Derivation of Theories for Displacement Based FEM from the New Variational Equation

4.1 Galerkin's approximations

Set $\boldsymbol{U}_h$, $\boldsymbol{S}_h$ and $\boldsymbol{E}_h$ to be finite dimensional subspaces of $\boldsymbol{U}$, $\boldsymbol{S}$ and $\boldsymbol{E}$, respectively. Define by $\boldsymbol{V}_h$ a natural product space of $\boldsymbol{U}_h$, $\boldsymbol{S}_h$ and $\boldsymbol{E}_h$. Then $\boldsymbol{V}_h\subset V$ and Galerkin's approximation on the space V_h to the solution of Eq. (27) is the function $(\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}) \in V_h$ such that

$$B(\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \nu, e, \tau) = l(\nu), \ \forall (\nu, e, \tau) \in V_h \quad (69)$$

If $\alpha = -\beta$, then Eq. (69) can be split as

$$\int_\Omega \sum_{i,j=1}^{3} \left(\sum_{k,l=1}^{3} C_{ijkl} u_{(k,l)} \right) \nu_{(i,j)} d\Omega + \beta \int_\Omega \sum_{i,j=1}^{3} \left(\sigma_{ij} - \sum_{k,l=1}^{3} C_{ijkl} \varepsilon_{kl} \right) \nu_{(i,j)} d\Omega = l(\nu), \ \forall \ \nu \in U_h \quad (70)$$

$$\int_\Omega \sum_{i,j=1}^{3} \left(\sum_{k,l=1}^{3} D_{ijkl} \sigma_{kl} \right) \tau_{ij} d\Omega = \int_\Omega \sum_{i,j=1}^{3} u_{(i,j)} \tau_{ij} d\Omega, \ \forall \ \tau \in S_h \quad (71)$$

$$\int_\Omega \sum_{i,j=1}^{3} \left(\sum_{k,l=1}^{3} C_{ijkl} \varepsilon_{kl} \right) e_{ij} d\Omega = \int_\Omega \sum_{i,j=1}^{3} \left(\sum_{k,l=1}^{3} C_{ijkl} u_{(k,l)} \right) e_{ij} d\Omega, \ \forall \ e \in E_h \quad (72)$$

In view of the definitions of S and E as given in Eqs. (11) and (12), one can aeeume $E_h = S_h$. Then there exists $e \in E_h$ such that

$$\tau_{ij} = \sum_{k,l=1}^{3} C_{ijkl} e_{kl} \quad (73)$$

for arbitrary $\tau \in S_h$. Substituting Eq. (73) into Eq. (71), one gets

$$\int_\Omega \sum_{i,j=1}^{3} \sigma_{ij} e_{ij} d\Omega = \int_\Omega \sum_{i,j=1}^{3} \left(\sum_{k,l=1}^{3} C_{ijkl} u_{(k,l)} \right) e_{ij} d\Omega, \ \forall \ e \in E_h = S_h \quad (74)$$

The right-hand-sides of Eqs. (72) and (74) are identical, and one can obtain

$$\int_\Omega \sum_{i,j=1}^{3} \sigma_{ij} e_{ij} d\Omega = \int_\Omega \sum_{i,j=1}^{3} \left(\sum_{k,l=1}^{3} C_{ijkl} \varepsilon_{kl} \right) e_{ij} d\Omega = \int_\Omega \sum_{i,j=1}^{3} \sigma^*_{ij} e_{ij} d\Omega, \ \forall \ e \in E_h = S_h \quad (75)$$

Since $\boldsymbol{\sigma}, \boldsymbol{\sigma}^* \in E_h = S_h$, the following equation is satisfied by Eq. (75):

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^*\|_0 = 0 \quad (76)$$

Applying Holder's inequality to the second integral of the right-hand-side of Eq. (70), one has

$$\left| \int_\Omega \sum_{i,j=1}^{3} \left(\sigma_{ij} - \sum_{k,l=1}^{3} C_{ijkl} \varepsilon_{kl} \right) \nu_{(i,j)} d\Omega \right| \le \|\boldsymbol{\sigma} - \boldsymbol{\sigma}^*\|_0 \|\nabla_s \nu\|_0 \le \|\boldsymbol{\sigma} - \boldsymbol{\sigma}^*\|_0 \|\nu\|_1, \ \forall \ \nu \in U_h \quad (77)$$

Since one can put, from Eqs. (76) and (77),

$$\int_\Omega \sum_{i,j=1}^{3} \left(\sigma_{ij} - \sum_{k,l=1}^{3} C_{ijkl} \varepsilon_{kl} \right) \nu_{(i,j)} d\Omega = 0, \ \forall \ \nu \in V_h \quad (78)$$

$$\int_\Omega \sum_{i,j=1}^{3} \left(\sum_{k,l=1}^{3} C_{ijkl} \varepsilon_{(k,l)} \right) \nu_{(i,j)} d\Omega = l(\nu), \ \forall \ \nu \in U_h \quad (79)$$

Furthermore, there exists $\tau \in S_h$ such that

$$e_{ij} = \sum_{k,l=1}^{3} D_{ijkl} \tau_{kl} \quad (80)$$

for arbitrary $e \in E_h$. Substituting Eq. (80) into Eq. (72), one finally obtains

$$\int_\Omega \sum_{i,j=1}^{3} \varepsilon_{ij} \tau_{ij} d\Omega = \int_\Omega \sum_{i,j=1}^{3} u_{(i,j)} \tau_{ij} d\Omega, \ \forall = \tau \in E_h \in S_h \quad (81)$$

Thus, the following theorem has been proved:

[Theorem 3] Let U_h, S_h and E_h be finite dimensional subspaces of U, S and E, respectively. Define by V_h the natural product space of U_h, S_h and E_h. If $\alpha = -\beta$ and $E_h = S_h$, then Galerkin's approximation on the space V_h to the solution of the variational equation given by Eq. (27) is the function $(\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}) \in V_h$ such that

$$\int_\Omega \sum_{i,j=1}^{3} \left(\sum_{k,l=1}^{3} C_{ijkl} u_{(k,l)} \right) \nu_{(i,j)} d\Omega = l(\nu), \ \forall \ \nu \in U_h \quad (82)$$

$$\int_\Omega \sum_{i,j=1}^{3} \sigma_{ij} e_{ij} d\Omega = \int_\Omega \sum_{i,j=1}^{3} \left(\sum_{k,l=1}^{3} C_{ijkl} u_{(k,l)} \right) e_{ij} d\Omega, \ \forall \ e \in E_h \quad (83)$$

$$\int_\Omega \sum_{i,j=1}^{3} \varepsilon_{ij} \tau_{ij} d\Omega = \int_\Omega \sum_{i,j=1}^{3} u_{(k,l)} \tau_{ij} d\Omega \ \ \forall \ \tau \in S_h \quad (84)$$

4.2 Stress/strain finite element equations of the unified variational equation

It is well known that Eq. (82) represents the variational state equation of displacement based FEM, while Eqs. (83) and (84) are the variational equations from which stress and strain solutions can be calculated from the displacement solution of Eq. (82). It shall now be shown that the finite element equations of Eqs. (83) and (84) are identical with the stress and strain smoothing equations of Hinton and Campbell (1974).

Since $E_h = S_h$, shape functions of stress and strain on a finite element space are identically defined. Therefore, one can define the Ith nodal shape function N^I of stress and strain within an element by the following function in terms of the natural coordinate system (ξ, η, α):

$$N^I = N^I(\xi, \eta, \zeta) \quad (85)$$

Then each component of stress and strain at any point within an element can be expressed by

$$\sigma_{ij}(\xi, \eta, \zeta)=\sum_{I=1}^{m} N^{I}(\xi, \eta, \zeta)\,\sigma_{ij}^{I}, \quad i, j=1, 2, 3 \tag{86}$$

$$\varepsilon_{ij}(\xi, \eta, \zeta)=\sum_{I=1}^{m} N^{I}(\xi, \eta, \zeta)\,\varepsilon_{ij}^{I}, \quad i, j=1, 2, 3 \tag{87}$$

where m denots the number of nodes per element, σ_{ij}^{I} and ε_{ij}^{I} denote the Ith nodal stress and strain, respectively. Denote by

$$\{\tilde{\sigma}_{ij}\}=[\sigma_{ij}^{1}, \sigma_{ij}^{2}, \cdots, \sigma_{ij}^{m}]^{T} \tag{88}$$

$$\{\tilde{\varepsilon}_{ij}\}=[\varepsilon_{ij}^{1}, \varepsilon_{ij}^{2}, \cdots, \varepsilon_{ij}^{m}]^{T} \tag{89}$$

the vectors comstructed from the ij components of nodal stress and strain.

Since the stress and strain functions are elements of $H^{0}(\Omega)$, Eqs. (83) and (84) are also valid within each element. Applying Eqs. (86) through (89) to Eqs. (83) and (84), one obtains the finite element equations of stress and strain components within an element as

$$[\boldsymbol{R}]\{\tilde{\sigma}_{ij}\}=\{\tilde{F}_{ij}\}, \quad i, j=1, 2, 3 \tag{90}$$

$$[\boldsymbol{R}]\{\tilde{\varepsilon}_{ij}\}=\{\tilde{G}_{ij}\}, \quad i, j=1, 2, 3 \tag{91}$$

where the element in the Ith row and Jth column of the smoothing matrix $[\boldsymbol{R}]$ and the Ith components of vectors of $\{\tilde{F}_{ij}\}$ and $\{\tilde{G}_{ij}\}$ are given as

$$R^{IJ}=\int_{\Omega_e} N^{I}N^{J}|\boldsymbol{J}|\,d\xi d\eta d\zeta \tag{92}$$

$$\tilde{F}_{ij}^{I}=\int_{\Omega_E} N^{I}\Big(\sum_{k,l=1}^{3} C_{ijkl}u_{(k,l)}\Big)|\boldsymbol{J}|\,d\xi d\eta d\zeta \tag{93}$$

$$\tilde{G}_{ij}^{I}=\int_{\Omega_e} N^{I}u_{(i,j)}|\boldsymbol{J}|\,d\xi d\eta d\zeta \tag{94}$$

where Ω_e is the domain of the element in the natural coordinate system and $|\boldsymbol{J}|$ is the determinant of the Jacobian matrix $\boldsymbol{J}$ relating the natural coordinate derivative to the global coordinate derivative.

Eqs. (90) and (91) are identical with the local stress and strain smoothing equations of Hinton and Campbell (1974). When shape functions of stress and strains are chosen such that $\sigma_{ij}\in \boldsymbol{H}^{1}(\Omega)$ and $\varepsilon_{ij}\in \boldsymbol{H}^{1}(\Omega)$, one obtains, in the a similar way, the stress and strain finite element equations defined on the whole domain. In this case, it can be easily shown that those finite element equations are identical with the global stress and strain smoothing equations of Hinton and Campbell (1974).

Thus, the following theorem has been proved :

[Theorem 4] The displacement based FEM using the stress/strain smoothing method of Hinton and Campbell is a form of mixed FEM based on the proposed variational principle.

5. Conclusion

The variational principle proposed in this paper provides a variational equation identical to the governing equations for the associated boundary value problem in linear elasticity. The energy bilinear form of the variational equation is symmetric, continuous and at least weakly- coercive. Therefore, there exists a unique solution of the variational equation by the generalized Lax-Milgram Theorem. Because of this feature, the variational principle can be used for the development of new improved finite element models.

It is shown that displacement based FEM using the stress/strain smoothing method of Hinton and Campbell (1974) can be derived as a special form of the mixed FEM based on the variational equation. Hence, this variational principle is a unified one which can provide a complete variational theory not only for mixed FEM, but also for displacement based FEM with the stress/strain smoothing method which was originally developed as a numerical technique lacking any variational basis.

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